UNIT V

18·8. SEQUENTIAL ANALYSIS - INTRODUCTION

We have seen that in Neyman-Pearson theory of testing of hypothesis, n, the sample size is regarded as a fixed constant and keeping α fixed, we minimise β. But in the sequential analysis theory propounded by *A. Wald* n, the sample size is not fixed but is regarded as a random variable whereas both α and β are fixed constants.

18·8·1. Sequential Probability Ratio Test (SPRT). The best known procedure in sequential testing is the *Sequential Probability Ratio Test* (SPRT) developed by A. Wald discussed below :

Suppose we want to test the hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$ for a distribution with p.d.f. $f(x, \theta)$. For any positive integer m, the likelihood function of a sample $x_1, x_2, \ldots, x_m$ from the population with *p.d.f.* (*p.m.f.*) $f(x, \theta)$ is given by :

$$L_{1m} = \prod_{i=1}^{m} f(x_i, \theta_1) \text{ when } H_1 \text{ is true, } \quad \text{and} \quad \text{by } L_{0m} = \prod_{i=1}^{m} f(x_i, \theta_0) \text{ when } H_0 \text{ is true,}$$

and the likelihood ratio λ_m is given by :

$$\lambda_m = \frac{L_{1m}}{L_{0m}} = \frac{\prod_{i=1}^{m} f(x_i, \theta_1)}{\prod_{i=1}^{m} f(x_i, \theta_0)} = \prod_{i=1}^{m} \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}, \ (m = 1, 2, \ldots) \qquad \ldots(18\cdot115)$$

The SPRT for testing H_0 against H_1 is defined as follows :

At each stage of the experiment (at the mth trial for any integral value m), the likelihood ratio λ_m, $(m = 1, 2, \ldots)$ is computed.

$$\left.\begin{array}{l}
(i) \text{ If } \lambda_m \geq A, \text{ we terminate the process with the rejection of } H_0 \\
(ii) \text{ If } \lambda_m \leq B, \text{ we terminate the process with the acceptance of } H_0, \text{ and} \\
(iii) \text{ If } B < \lambda_m < A, \text{ we continue sampling by taking an additional observation.}
\end{array}\right\} \qquad \ldots(18\cdot116)$$

Here A and B $(B < A)$ are the constants which are determined by the relation

$$A = \frac{1-\beta}{\alpha}, \qquad B = \frac{\beta}{1-\alpha} \qquad \ldots(18\cdot117)$$

where α and β are the probabilities of type I error and type II error respectively.

From computational point of view, it is much convenient to deal with $\log \lambda_m$ rather than λ_m, since

$$\log \lambda_m = \sum_{i=1}^{m} \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} = \sum_{i=1}^{m} z_i \qquad \text{...(18·118)}$$

where $$z_i = \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} \qquad \text{...(18·118a)}$$

In terms of z_i s, *SPRT* is defined as follows :

(*i*) If $\Sigma z_i \geq \log A$, reject H_0

(*ii*) If $\Sigma z_i \leq \log B$, reject H_1 (Accept H_0) ...(18·119)

(*iii*) If $\log B < \Sigma z_i < \log A$, continue sampling by taking an additional observation.

Remarks. 1. In *SPRT*, we continue taking additional observations unless the inequality

$$B < \lambda_m < A \Rightarrow \log B < \Sigma z_i < \log A,$$

is violated at either end. It has been proved that SPRT eventually terminates with probability one.

2. Sequential schemes provide for a minimum amount of sampling and thus result is considerable saving in terms of inspection, time and money. As compared with single sampling, sequential scheme requires on the average 33% to 50% less inspection for the same degree of protection *i.e.*, for the same values of α and β.

18·8·2. Operating Characteristic (O.C.) Function of SPRT. The O.C. function $L(\theta)$ is defined as

$L(\theta)$ = Probability of accepting $H_0 : \theta = \theta_0$ when θ is the true value of the parameter.

and since the power function

$P(\theta)$ = Probability of rejecting H_0 where θ is the true value, we get

$$L(\theta) = 1 - P(\theta) \qquad \text{...(18·120)}$$

The O.C. function of a SPRT for testing $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$, in sampling from a population with density function $f(x, \theta)$ is given by

$$L(\theta) = \frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}} \qquad \text{...(18·121)}$$

where for each value of θ, the value of $h(\theta) \neq 0$ is to be determined so that

$$E\left[\frac{f(x, \theta_1)}{f(x, \theta_0)}\right]^{h(\theta)} = 1 \qquad \text{...(18·122)}$$

where the constant A and B have already been defined in (18·117). It has been proved that under very simple conditions on the nature of the function $f(x, \theta)$, there exists a unique value of $h(\theta) \neq 0$ such that (18·122) is satisfied.

18·8·3. Average Sample Number (A.S.N.). The sample size n in sequential testing is a random variable which can be determined in terms of the true density function $f(x, \theta)$. The A.S.N. function for the S.P.R.T. for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ is given by

$$E(n) = \frac{L(\theta) \log B + [1 - L(\theta)] \log A}{E(Z)} \qquad \text{...(18·123)}$$

where $$Z = \log\left(\frac{f(x, \theta_1)}{f(x, \theta_0)}\right),\ A = \frac{1-\beta}{\alpha},\ B = \frac{\beta}{1-\alpha} \qquad \text{...(18·123a)}$$

$f(x,\theta) = \theta^{x} (1-\theta)^{1-x} \qquad x = 0,1$

for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ construct SPRT.

solution. Given $f(x,\theta) = \theta^{x}(1-\theta)^{1-x}$

$$\lambda_m = \frac{L_{1m}}{L_{0m}} = \frac{\theta_1^{\sum_{i=1}^{m} x_i} (1-\theta_1)^{\sum(1-x_i)}}{\theta_0^{\sum_i x_i} (1-\theta_0)^{\sum(1-x_i)}}$$

$$= \left(\frac{\theta_1}{\theta_0}\right)^{\sum_{i=1}^{m} x_i} \left(\frac{1-\theta_1}{1-\theta_0}\right)^{m-\sum_{i=1}^{m} x_i}$$

$$\log \lambda_m = \sum_{i=1}^{m} x_i \log\left(\frac{\theta_1}{\theta_0}\right) + \left(m - \sum_{i=1}^{m} x_i\right). \log\left(\frac{1-\theta_1}{1-\theta_0}\right)$$

$$= \sum_{i=1}^{m} x_i (\log\theta_1 - \log\theta_0) + m.\log\left(\frac{1-\theta_1}{1-\theta_0}\right) - \sum_{i=1}^{m} x_i \left(\log(1-\theta_1) - \log(1-\theta_0)\right)$$

$$= \sum_{i=1}^{m} x_i \log\left\{\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right\} + m\log\left(\frac{1-\theta_1}{1-\theta_0}\right)$$

Hence SPRT for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ is as follows.

1. Accept H_0 if

$$\log \lambda_m \le \log\left(\frac{\beta}{1-\alpha}\right)$$

$$\sum_{i=1}^{m} x_i \log\left\{\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right\} + m\log\left(\frac{1-\theta_1}{1-\theta_0}\right) \le \log\left(\frac{\beta}{1-\alpha}\right)$$

$$\sum_{i=1}^{m} x_i \log\left\{\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right\} \le \log\left(\frac{\beta}{1-\alpha}\right) - m\log\left(\frac{1-\theta_1}{1-\theta_0}\right)$$

$$\Rightarrow \sum_{i=1}^{m} x_i \le \frac{\log\left(\frac{\beta}{1-\alpha}\right) - m\log\left(\frac{1-\theta_1}{1-\theta_0}\right)}{\log\left\{\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right\}} \quad \text{— A } H_0$$

2. Reject H_0 if

$$\log \lambda_m \ge \log\left(\frac{1-\beta}{\alpha}\right)$$

(2)

$$\Rightarrow \sum_{x=0}^{1} \left\{ \left(\frac{\theta_1}{\theta_0}\right)^x \left(\frac{1-\theta_1}{1-\theta_0}\right)^{1-x} \right\}^{h(\theta)} \cdot \theta^x (1-\theta)^{1-x} = 1$$

$$\Rightarrow \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)} (1-\theta) + \left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} \cdot \theta = 1$$

$$\Rightarrow \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)} - \theta \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)} + \theta \cdot \left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} = 1$$

$$\Rightarrow \theta \left\{ \left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} - \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)} \right\} = 1 - \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)}$$

$$\Rightarrow \theta = \frac{1 - \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)}}{\left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} - \left(\frac{1-\theta_1}{1-\theta_0}\right)^{h(\theta)}} \quad \text{— (1)}$$

we have

$$L(\theta) = \frac{A^{h(\theta)} - 1}{A^{h(\theta)} - B^{h(\theta)}}$$

$$= \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h} = L(\theta, h) \quad \text{— (2)}$$